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A Spherical Cavity in an Einstein Universe

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Suitable metric forms for the regions inside and outside a spherical cavity in an Einstein universe are derived by means of perturbation. It is shown that for low proper pressure, the cavity behaves like "negative" Schwarzschild mass. Finally, the possibility of carrying over to the exact theory a proposed definition of the gravitational field in matter is examined.

1. INTRODUCTION

In a previous paper (Kofinti, 1980), we proposed a definition in general relativity for the linear gravitational field in the interior of a continuous distribution of matter. The proposed definition is based on the technique of introducing a small fictitious cavity inside matter and proceeding to the limit when the cavity is vanishingly small.

It should be remarked that the idea of introducing a cavity in a Riemannian space-time manifold is not altogether foreign to the literature in general relativity. For example, by considering a hollow Schwarzschild sphere and a hollow static torus, Marder (1964) came to the conclusion that in many cases a spatially bounded empty region S in a space-time may be replaced by any class C^2, piecewise class C^4 diffeomorphic space-time region S' if the bounding matter distribution is modified to preserve the continuity conditions of Lichnerowicz (1955). We mention also the "Swiss cheese model," which is a Friedmann world with an arbitrary number of spherically symmetric condensations situated in the centers of nonoverlapping spherical holes which have Schwarzschild's metric, Heckmann and Schücking (1962); see also Bonnor (1956), and Einstein and Straus (1945).

In this paper, the possibility of carrying over the proposed definition of the gravitational field to the exact theory is studied by introducing a

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spherical cavity in an Einstein universe. The appropriate metric forms are derived in Section 2 and these are matched across the junction in Section 3. In Section 4, a Newtonian analog is derived by considering the special case of low proper pressure, while in Section 5 the gravitational field in matter is examined by considering the limiting behavior of the Weyl tensor in the cavity introduced. Section 6 contains some discussion and conclusions.

2. DERIVATION OF METRIC FORMS

Consider a small empty spherical region (II) introduced in a static Einstein universe. Owing to the homogeneity of the Einstein universe, we can take the cavity as centered at the origin $r=0$ of the instantaneous 3-space. However, owing to the unstable nature of the Einstein universe, we introduce fictitious stresses in the nonempty region (I) outside the cavity so as to hold up the matter, as it were, from expanding away to infinity.

We take for the empty region (II) inside the cavity the spherically symmetric metric form

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varnothing^2) \tag{2.1}$$

where λ, ν are functions of r only and the velocity of light in vacuo is taken as unity (i.e., $c=1$). Now, since the region (II) is empty, the corresponding energy-momentum tensor $\overset{(\mathrm{II})}{T_{ij}}$ vanishes and so the field equations

$$R_j^i - \tfrac{1}{2}\delta_j^i R + \delta_j^i \Lambda = -8\pi T_j^i \tag{2.2}$$

(where Λ is the cosmological constant) and (2.1) lead to a solution of the form

$$ds^2 = q(r)dt^2 - dr^2/q(r) - r^2(d\theta^2 + \sin^2\theta\, d\varnothing^2) \tag{2.3}$$

where $q(r) = 1 + \alpha/r - \Lambda r^2/3$, and α is an arbitrary constant of integration.

In the case of the nonempty region (I) outside the cavity, we regard the metric as a first-order perturbation of the original Einstein universe which is given by the metric form

$$ds_0^2 = dt^2 - dr^2/q_0(r) - r^2(d\theta^2 + \sin^2\theta\, d\varnothing^2) \tag{2.4}$$

where $q_0(r) = 1 - r^2/R_0^2$, $1/R_0^2 = \Lambda - 8\pi p_0$, where p_0 is the proper pressure.

Suppose the small spherical cavity has a (small) radius ε. Then, from Newtonian considerations, we expand the field variables to order ε^3. Thus, for region (I), we write

$$e^{-\lambda} \equiv \sum_{n=0}^{3} \varepsilon^n f_n(r), \qquad e^{\nu} \equiv \sum_{n=0}^{3} \varepsilon^n g_n(r), \qquad p(r) \equiv \sum_{n=0}^{3} \varepsilon^n p_n(r) \quad (2.5)$$

where $f_n(r), g_n(r), p_n(r)$ are functions of r only and f_0, g_0, p_0 refer to the original unperturbed metric in (2.4).

Assuming spherical symmetry after the perturbation, (2.1) and (2.2) yield the usual equations

$$\begin{aligned} e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 + \Lambda &= 8\pi p_0 \\ e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 - \Lambda &= 8\pi\rho_0 \\ p_0' + (\rho_0 + p_0)\nu'/2 &= 0 \end{aligned} \quad (2.6)$$

where p_0, ρ_0 are, respectively, proper pressure and density and a prime denotes differentiation with respect to r. We now substitute the expansions (2.5) into (2.6) and retain terms up to order ε^3. On equating to zero the sum of all terms that are independent of ε, we then obtain the relations

$$\begin{aligned} g_0' f_0 + f_0 g_0/r^2 + (\Lambda - 1/r^2) g_0 - 8\pi g_0 p_0 &= 0 \\ f_0'/r + f_0/r^2 + (\Lambda - 1/r^2) + 8\pi\rho_0 &= 0 \\ p_0' g_0 &= 0 \end{aligned} \quad (2.7)$$

which are indeed satisfied by the original unperturbed Einstein universe since (2.4) gives

$$f_0(r) \equiv 1 - r^2/R_0^2, \qquad g_0(r) \equiv 1, \qquad 8\pi p_0 = \Lambda - 1/R_0^2 \quad (2.8)$$

On equating to zero the coefficients of terms in ε and solving the resulting differential equations, we obtain

$$f_1(r) = A/r, \qquad g_1(r) = B - AR_0^2/r^3, \qquad p_1(r) = A/8\pi r^3 \quad (2.9)$$

where A, B are arbitrary constants of integration. Continuing the above procedure, we obtain from ε^2 terms

$$f_2(r) = C/r, \qquad g_2(r) = D - (AB + C)R_0^2/r^3 + R_0^4 A^2/r^6, \qquad p_2(r) = C/8\pi r^3 \quad (2.10)$$

while ε^3 terms yield

$$f_3(r) = E/r$$

$$g_3(r) = k^2\Big[\tfrac{1}{3}ar^5 - br^2\ln r - 3ak^2r^3 - 3ak^4r - \tfrac{3}{2}bk^2 + (ak^6 - c)/3r + 3bk^4/r^2 + 3ck^2/5r^3 + F - bk^6/6r^4 - 3ck^4/7r^5 + ck^6/9r^7\Big]/(r^2-k^2)^4 \quad (2.11)$$

$$8\pi p_3(r) = E/r^3 + (\Lambda - 1/r^2)g_3$$

where C, D, E, F are arbitrary constants of integration and we have put

$$k^2 \equiv 1/(\Lambda + 1/R_0^2), \qquad a \equiv 3(E + BC)$$

$$b \equiv 6AR_0^2(AB + C), \qquad c = 9R_0^4A^3 \quad (2.12)$$

The metric form appropriate for the nonempty region (I) outside the cavity is thus given by (2.1) with e^ν, e^λ given by (2.5) and (2.9)–(2.12).

3. MATCHING OF THE SOLUTIONS

We now determine the arbitrary constants of integration occurring above by applying suitable boundary conditions to the metric forms (2.3) for region (II) and (2.4), (2.5), (2.9)–(2.12) for region (I). From the continuity of the metrics across the boundary surface $r = \varepsilon$ of the cavity we obtain

$$\alpha = A = C = 0, \qquad E = 1/R_0^2 - \Lambda/3$$

$$B = F = 0, \qquad D = -1/R_0^2 \quad (3.1)$$

Indeed, we could have put the arbitrary constant α equal to zero, ab initio, so as to ensure the regularity of the metric form (2.3) inside the cavity. On using (3.1) in (2.12), we find

$$a = 3E = 3/R_0^2 - \Lambda (= 8\pi\rho_0), \qquad b = c = 0 \quad (3.2)$$

and hence from (2.11) we obtain

$$g_3(r) = 8\pi\rho_0 k^2\Big[\tfrac{1}{3}r^5 - 3k^2r^3 - 3k^4r + k^6/3r\Big]/(r^2-k^2)^4 \quad (3.3)$$

where k^2 is given by (2.12).

It follows from (2.5), (2.8)–(2.12), and (3.3) that the metric form appropriate for the perturbed Einstein universe exterior to the cavity [i.e., region (I)] is given by

$$dS_{\mathrm{I}}^2=\left[1+\frac{6Mk^2}{(r^2-k^2)^4}(r^5/3-3k^2r^3-3k^4r+k^6/3r)\right]dt^2$$
$$-\left(1-r^2/R_0^2+2M/r\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\varnothing^2) \qquad (3.4)$$

where k^2 is given by (2.12) and we have put $M\equiv\frac{4}{3}\pi\varepsilon^3\rho_0$. It will be noticed from (3.4) that as we shrink the cavity to zero (i.e., as ε, or M, tends to zero) we recover the original unperturbed metric form in (2.4).

Furthermore, from (2.3) and (3.1), the metric form inside the cavity [region (II)] is now

$$dS_{\mathrm{II}}^2=(1-\Lambda r^2/3)dt^2-(1-\Lambda r^2/3)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\varnothing^2) \quad (3.5)$$

which is an empty de Sitter universe.

4. CASE OF LOW PRESSURE

In this section, we consider the special case of an Einstein universe with low proper pressure p_0. In this case, (2.8) and (2.12) imply that we can take $k^2=1/2\Lambda$. Substituting this value of k into the expression (3.3) for $g_3(r)$ and expanding up to first order in Λ, we obtain

$$g_3(r)=8\pi\rho_0/3r-80\pi\rho_0\Lambda r/3 \qquad (4.1)$$

It follows from (2.5) and (4.1) that the g_{00} potential outside the cavity is given by

$$\begin{aligned} e^{\nu}&=1+8\pi\rho_0\varepsilon^3/3r\\ &=1+2M/r \end{aligned} \qquad (4.2)$$

on neglecting the product $\Lambda\varepsilon^3$ (in accordance with our approximations) and putting $M=\frac{4}{3}\pi\varepsilon^3\rho_0$, where M is the proper mass of the material originally occupying the small cavity introduced.

We see from (4.2) that as far as the g_{00} potential outside the cavity is concerned, the empty cavity we have introduced behaves like a "negative" Schwarzschild mass located at the spatial origin $r=0$. This is in remarkable agreement with the corresponding Newtonian result and so justifies once more the usual identification of the time-time component of the weak field metric with the Newtonian potential.

5. BEHAVIOR OF THE WEYL TENSOR

In this section, we examine the behavior of the Weyl tensor (i.e., the gravitational field) in the (small) cavity as we shrink the latter to zero. Using the metric form in (3.5), we obtain by direct computation the following surviving components of the Weyl tensor:

$$\begin{aligned} C_{2020} &= -C_{1212} = \Lambda r^2/3 \\ C_{3030} &= -C_{3131} = \Lambda r^2 \sin^2\theta/3 \\ C_{1010} &= \Lambda/3 \end{aligned} \tag{5.1}$$

Hence, as we shrink the cavity to zero, all the components of the Weyl tensor vanish except the component

$$C_{1010} = \Lambda/3 \tag{5.2}$$

On the other hand, we find from (2.4) that prior to the introduction of the cavity, the only surviving components of the Weyl tensor are given by

$${}^0C_{1212} = \operatorname{cosec}^2\theta \; {}^0C_{3131} = -2\Lambda r^2 \tag{5.3}$$

which clearly vanish at the spatial origin $r = 0$.

It follows from (5.2) and (5.3) that for a nonzero cosmological constant, the limit of the Weyl tensor in the cavity, as we shrink the cavity to zero, is not equal to the original value (at $r = 0$) in the unperturbed Einstein universe.

6. CONCLUSIONS

In defining the gravitational field in the exact theory of general relativity, the linear considerations given in a previous paper (Kofinti, 1980) would suggest cutting small cylindrical pipes. As remarked by Wheeler (1965), the pipe is an idealized device introduced to separate the effects of gravitational forces from those of pressure. However, Newtonian theory suggests that the statistical condition limits the equilibrium forms of fluids to spherical symmetry. Hence we cannot expect to find in general relativity a statical fluid body which is axially symmetric without being spherically symmetric (Synge, 1960).

The results in Section 5 of this paper suggest that except in the case of a negligible cosmological constant the definition proposed previously (Kofinti, 1980) requires some modification in the exact theory.

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